NUMERICAL SIMULATION OF FRACTURE IN CONCRETE USING PHASE FIELD MODEL

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Abstract- In this paper, a phase field-based numerical model has been presented to study fracture in concrete. All nonlinearities in the fracture process zone are modeled using cohesive zone approach with traction-separation constitutive law for concrete. A localized band of finite width is used to regularize the crack path using scalar phase-field. The phase-field discriminate between intact and broken surface using numeric values of 1 and 0. The critical fracture energy is modeled as the algebraic sum of the critical fracture energy of mode I and mode II. A finite element method is used to implement the proposed model. Numerical simulation for mode I fracture is performed on a three-point notched concrete beam. The concrete response under applied loading is represented using load-displacement curve. A study related to effect of mesh size and length scale parameter on the output results is carried out. The results indicated that the length scale parameter and finite element size has little effect on the model. It is concluded that the phase-field model has the ability to simulate crack growth in concrete under the given loading condition.

Keywords- Mode I fracture, Mixed mode fracture, Phase-field, Fracture energy, Length scale parameter.

1 Introduction

Concrete is composite material and is widely used in construction industry as a building material. When subjected to normal and shear stresses, the crack tends to propagates along an inclined path, indicating the mixed mode fracture in concrete. The mixed mode fracture in concrete is one which occurs as result of the combination of mode I (tensile failure) and mode II (shear failure). Various experimental tests have shown that concrete generally fails in mixed mode (I-II) when subjected to environmental loading [1]. Understanding the fracture phenomena in concrete is vital for the accurate analysis and design of concrete structure. This is because the cracking reduces the stiffness of concrete, thereby effecting its serviceability, durability and may lead to complete failure. A numerical model generally needs two ingredients to model crack growth problem in concrete: the first one is the crack initiation criterion and the second one is the damage criterion. The crack initiation criterion is generally based on maximum principle tensile stress or strain. Other crack initiation criterion can also be used. The damage criterion has to be based on mode I and mode II fracture energies if one wants to study the crack growth under mixed mode condition. Developing these criteria for mixed mode fracture is challenging [2]. Although, most of the numerical model, which are proposed earlier [3], [4], can simulate the crack initiation and crack propagation, they are unable to simulate some aspects of crack related to fracture process, for example crack kinking, coalescence, branching and nucleation. The phase field model presented in this paper has the ability to predict the crack initiation, propagation, nucleation, branching and kinking during crack growth.

The macro model behaviors of concrete structure can be determined from the meso or micro model if mechanical properties of the two matches [5]. In addition, performing experiments on macro structures is time consuming and uneconomical. For this reason, meso and micro scale model of the actual physical structures are generally used for simulation. To achieve better results, greater degree of mechanical, geometrical and compositional similarity between the prototypes and actual system is required [6].

Two approaches are generally used to model the fracture in concrete: discontinuous and continuous approach. The discontinuous approach is largely based on the work presented in [7], [8] and Cohesive Zone Model (CZM) [9], [10]. In this approach the crack is modeled as a strong discontinuity embedded in the finite element mesh. In contrast, the continuous approach models the crack by distributing the damage in a band of finite width using a scalar phase field.

The Cohesive Zoned Model and Extended Finite Element Method (XFEM) are the two prominent methods which are based on the discontinuous approach. In CZM, the cohesive elements are distributed along the boundary of the element in finite element mesh. A mechanical behavior of cohesive element is controlled by traction separation law, that also account for all nonlinearities at the crack tip. Many researchers [11]–[13] have used this approach to simulate the fracture phenomena in concrete and other material. This approach requires the crack path to be known in advance. In addition, this method is mesh sensitive [14]. The complicated stress distribution in the fracture process zone cannot be accounted for in this method [15]. This method requires advance software and packages for implementation. As opposed to CZM, the XFEM is free from mesh sensitivity problems and can easily handle the displacement discontinuity due to sharp crack [16]. This method is based on partition of unity [17]. The discontinuity is included in the finite element mesh by enriching specific node using special function without altering the original mesh. XFEM has the ability to simulate complex phenomena and more powerful as compare to CZM. Numerous researcher have used this method for simulating and numerical modeling of complex problems like fracture in concrete, see for example [18]–[22]. Although, XFEM is a powerful numerical method, it too like CZM suffer in cases of complex crack topologies including crack branching, intersecting and kinking. It is also not suitable to simulate problem involving friction action and high degree of material nonlinearities.

The continuous approach, as opposed to discontinuous approach, does not model the crack as a strong discontinuity rather smears it in a localized band. The phase field model (PFM) falls in this category. In this model, the damage is distributed in a localized band and a scalar phase field ϕ is used to indicate the degree of damage in the material. The width of the damage is controlled by a length scale parameter *lc.* The phase field take value of 0 and 1 for completely intact and damage state of the material. The PFM gives the displacement and phase field as output of the model by solving multivariable problem. It do so by fusing the Griffith criterion and phase field in the total potential of the system and extremizing it using the variational principle [23]. The phase field, surface density and damage equation used in the PFM can vary and thus results in different phase field models. Feng and Wu [24] used a phase field model to investigate the boundary and size effect on fracture in concrete. The cohesive zone model approach was used to model the strain softening behavior of concrete. In [25], the phase field model was used to study fracture in cement-based material. A modified constitutive law inclusive of the early-age processes including shrinkage, thermal expansion and creep was used to simulate fracture. In addition, ,many other researcher [26]–[30] have used phase field model to simulate the fracture process in solids.

In this paper, phase field model proposed in [30] has been used to simulate mode I fracture in simply supported notched concrete beam. The load is applied at mid-point of the beam to ensure that the crack grow in mode I. The aim of the study is to investigate the suitability of phase field model in simulating fracture in concrete. In addition, the effect of length scale parameter on the crack growth has been undertaken. The concrete mass is assumed to be homogenous in composition. All nonlinearities are included in the model using an exponential traction-separation law. No viscous parameter is included in the model. The simulations are performed using the elastic mechanical properties of the material. The crack and damage visualization in the model during simulating is assisted by the scalar phase field discriminating between intact and damage state of the material using numerical value of 0 and 1.

2 Phase Field Model

In this section, the formulation of phase field model based on the variation approach has been presented. For a solid body with domain Ω, the external potential energy can be written as:

$$P(\mathbf{u}) = \int_{\Omega} b.\mathbf{u}dV + \int_{\partial\Omega} t.\mathbf{u}dA \quad (1)$$

Here b is the body force, t is the traction applied on $\partial\Omega_t$ and $\boldsymbol{u}$ is the displacement applied at the boundary.

For a crack set Γ in the solid domain Ω ,the expression for the surface energy is given by:

$$\Psi(\Gamma) = \int_{\Gamma} G_c dA \quad (2)$$

The G_c is the critical energy release rate of the material. it can be obtained from tensile test on the material.

The internal strain energy stored in the solid body can be represented as following:

$$\Psi_0(u) = \frac{1}{2}\lambda tr^2[\varepsilon] + \mu tr[\varepsilon^2] \quad (3)$$

Where λ and μ are the lame constant and ε is the strain tensor.

After having equations for the external potential energy, surface energy and strain energy density function, the total potential energy of system, ignoring the body forces, can be written as:

$$\Pi(\mathbf{u},\phi) = \int_{\Omega} \Psi(\varepsilon(\mathbf{u}))dV + \int_{\Gamma} G_c dA - \int_{\partial\Omega} t.\mathbf{u}dA \quad (4)$$

The numerical implementation of the equation (4) is a challenge because of the discontinuous displacement field and unknow nature of the crack set. These limitation are addressed by regularizing the total potential of the system using variation approach presented in [26], [27], [31]. The regularized total potential is given by:

$$\Pi(\mathbf{u},\phi) = \Psi_s(\mathbf{u},\phi) + \Psi_c(\Gamma) - P(\mathbf{u}) \quad (5)$$

Where

$$\Psi_s(\mathbf{u},\phi) = \int_{\Omega} g(\phi)\psi_0(\grave{o}(\mathbf{u}))dV \quad (6)$$

$$\Psi_c(\Gamma) = \int_{\Omega} G_c \gamma(\phi;\nabla\phi)dA \quad (7)$$

Here $\Psi_s(\mathbf{u},\phi), \Psi_c(\Gamma)$ is the regularized strain energy density functional and regularized surface energy functional. $\gamma(\phi;\nabla\phi)$ is the crack surface density functional characterizing the growth of phase field in the solid domain. $P(\mathbf{u})$ is the external potential energy due to body forces and traction.

The crack surface density, $\gamma(\phi;\nabla\phi)$ function is given by, [26], [32]:

$$\gamma(\phi;\nabla\phi) = \frac{1}{2}\left[\frac{1}{l_0}\phi^2 + l_0(\nabla\phi\cdot\nabla\phi)\right] \quad (8)$$

After putting Equation (6), (7) and (8) in Equation (5), the total potential can be rewritten as, [30] :

$$\Pi(\mathbf{u},\phi) = \int_{\Omega} g(\phi)\psi_0(\grave{o}(\mathbf{u}))dV + \int_{\Omega} G_c \frac{1}{2}\left[\frac{1}{l_0}\phi^2 + l_0(\nabla\phi\cdot\nabla\phi)\right]dV - \int_{\Omega} b.udV - \int_{\partial\Omega} t.udA \quad (9)$$

In equation (9) $\psi_0(\grave{o}(\mathbf{u}))$ is the elastic free energy functional and $g(\phi)$ is the damage function. The damage function is bound to the following conditions, [26]:

$$g(0)=0, \quad g(1)=1 \ \ and \ \ \dot{g}(1)=0 \quad (10)$$

After expressing the total potential of the solid body in term of phase field model, the problem comes down to finding the variation of Equation (9) w.r.t phase field ϕ and displacement u given by:

$$\delta\Pi\left(\mathbf{u},\phi\right)=\int_{\Omega}\sigma:\delta\varepsilon dV+\int_{\Omega}\frac{\partial\psi}{\partial\phi}\delta\phi dV+\int_{\Omega}G_c\left(\frac{\partial\gamma}{\partial\phi}+\frac{\partial\gamma}{\partial\nabla\phi}.\delta\nabla\phi\right)dV-\int_{\Omega}b.\delta\mathbf{u}dV-\int_{\partial\Omega_t}t.\delta\mathbf{u}dA \tag{11}$$

It yields the following equations and boundary conditions, [30]:

$$\nabla\cdot\sigma+b=0 \quad in\,\Omega \tag{12}$$

$$\sigma\cdot n=t \quad on\ \partial\Omega_t \tag{13}$$

$$Y-G_c\partial_{\phi}\gamma=0 \qquad \dot{\phi}>0 \qquad on\ \Omega \tag{14}$$

$$Y-G_c\partial_{\phi}\gamma=0 \qquad \dot{\phi}=0 \qquad on\ \Omega \tag{15}$$

Where

$$Y:=-\frac{\partial\psi}{\partial\phi}$$

$$\partial_{\phi}\gamma:=\frac{\partial\gamma}{\partial\phi}-\nabla\cdot\left(\frac{\partial\gamma}{\partial\nabla\phi}\right) \tag{16}$$

$$\frac{\partial\gamma}{\partial\nabla\phi}.n_B=0 \quad on\ \partial B \tag{17}$$

Equation (12) is called the equilibrium equation while Equation (14) and (15) are the phase field evolution equation. In addition, equation (13) and (17) are the boundary condition.

In order to simulate the mixed mode fracture and avoid the crack growth under compression, the initial strain energy is decomposed into tensile, shear and compression part as following, [30]:

$$\psi_0=g_1\left(\phi\right)\psi_{0I}^{+}+g_{II}\left(\phi\right)\psi_{0II}^{+}+\psi_0^{-} \tag{18}$$

$$\psi_0^{-}=\psi_0-\psi_{0I}^{+}-\psi_{0II}^{+} \tag{19}$$

The mixed mode phase field evolution equation becomes:

$$g_I'\left(\phi\right)\frac{\psi_{0I}^{+}}{G_{cI}}+g_{II}'\left(\phi\right)\frac{\psi_{0II}^{+}}{G_{cII}}\geq\frac{2l_0^2\nabla^2\phi-\alpha'\left(\phi\right)}{c_0l_0} \tag{20}$$

Where g_I' and g_{II}' are the degradation function for mode I and mode II. Similarly, G_{cI} and G_{cII} are the fracture energies for the two modes.

The weak form of the governing differential equation is given by:

$$\int_{\Omega}B_u^T\hat{D}B_u\,\partial\Omega\,\hat{u}=\int_{\partial\Omega}\Phi_u^T\,\bar{t}\,d\partial\Omega \tag{21}$$

$$\int_{\Omega}\frac{2l_0}{c_0}B_s^TB_s\,d\Omega\,\hat{\phi}+\int_{\Omega}\frac{1}{c_0l_0}\alpha\left(\phi\right)\Phi_{\phi}^Td\Omega+\int_{\Omega}\left[\frac{g_I'\left(\phi\right)}{b_I}h_1\right]+\left[\frac{g_{II}'\left(\phi\right)}{b_{II}}h_{II}\right]\Phi_{\phi}^Td\Omega=0 \tag{22}$$

Where $\hat{u}$ and $\hat{\phi}$ are the known displacement and phase field values at nodes. $\hat{D}$ is the modified constitutive matrix. Φ_u and Φ_{ϕ} are shape functions for displacement and phase field. The b_I and b_{II} are constants that can affect fracture angles during crack growth. The phase field evolution equation and equilibrium equation are solved for displacement and phase field and the crack path is obtain as part of the solution. The propose scheme is implemented in a finite element code using C++ Jem and Jive libraries

3 Methodology

The methodology used for simulating fracture in concrete using the proposed phase field model is discussed in the following sections.

3.1 Specimen Selection

The concrete beam specimen used in [33] has been selected for numerical simulation of mode I fracture in concrete using the proposed phase field model. The dimensions of the beam are: length, L = 36 in, Depth, D = 9 in and thickness, t = 3.375 in. However, to reduce the computational cost, the dimensions are taken in millimeter (mm). The beam has a notch of depth 3 mm in the middle on the lower face as shown in the Figure 1.

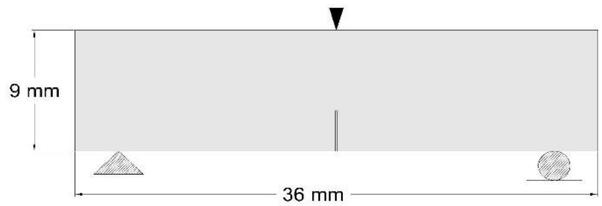


Figure 1.Geometry of notched concrete beam with boundary condition

3.2 Numerical Simulation of Three-Point Beam

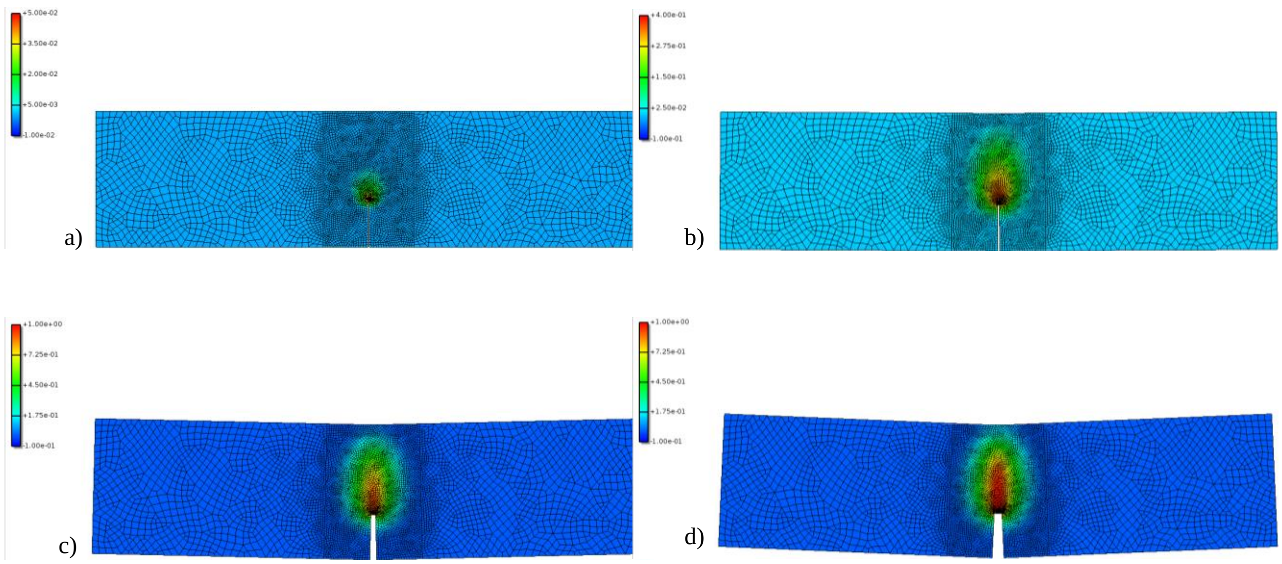
The material properties required in the model include modules of elasticity E, the Poisson's ratio υ, critical fracture energy G_f, the ultimate tensile strength f_t. For the given beam, the fracture energies of mode I and mode II are taken to be equal. Such assumption can be made because the scope of the study conducted in this paper is limited only to simulating the fracture process. More specifically, the ability of the model to produce the generic load-displacement curve of concrete is the center of focus. There is no validation work to be carried out. The problem is considered as a plane strain problem and exponential cohesive law has been used. The initial notched is modelled as a discrete crack in the beam.

The beam has length of 36 mm and depth of 9 mm. The notch has depth of 3 mm and is located in the center of beam. The material properties of the beam are taken from [13] and include modulus of elasticity E = 142000 N/mm^2, Poisson's ratio υ = 0.35, fracture energy G_f = 0.344 N/mm, tensile strength f_t = 3.4 N/mm^2 and shear strength f_s = 3.4 N/mm^2. The fracture energy of mode I has deliberately been taken equal to mode II fracture energy i.e., $G_I = G_{II} = G_f$. The simulation is performed under displacement scheme with load scale of 1.0 e^{-3} in 900 iterations. The beam domain is discretized using 6037 quadrilateral elements as shown in Figure 2. In order to reduce the computation time, the Dirichlet boundary condition $\phi = 0$ has been assign to the elements close to the support.

Figure 2.Descritized geometry of the notched concrete beam

4 Results and Discussion

The first trial of simulation is performed using mesh size of 0.3 mm in the region close to the crack and 0.5 mm in the remaining region. It can be observed that during the initial iteration the stress at the crack tip increases. As a result, the phase field values increase and upon reaching 1, the element damages completely and the crack progress further. The crack path, as shown in the Figure 3, propagates vertically indicating the mode I fracture. This can be attributed to the fact that



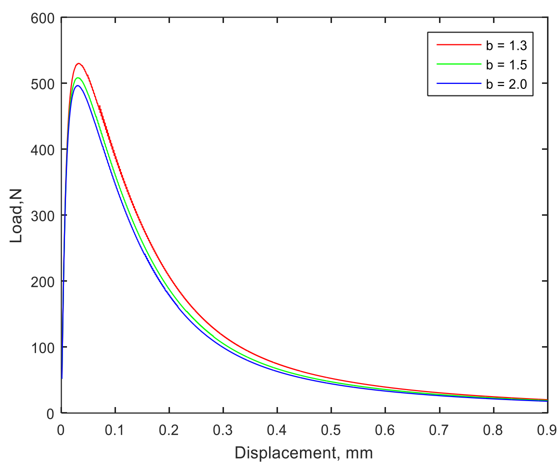


Figure 4. Load-Displacement curve for length scale parameter of b=1.3 mm, b=1.5 mm and b=2.0 mm

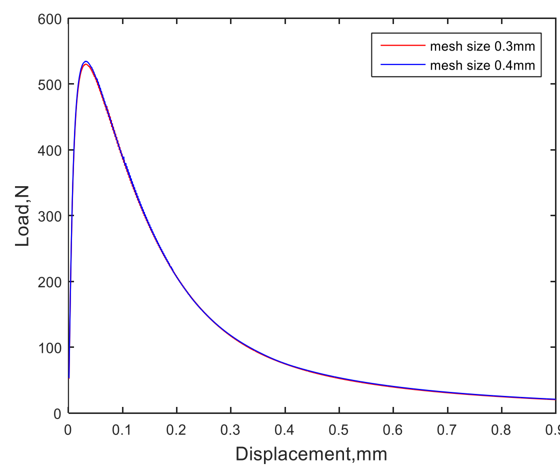


Figure 5. Load-Displacement curve for mesh size of 0.3 mm and 0.4 mm with constant b = 2 mm

Figure 3.Crack propagation in notched concrete beam under three-point bending, a. crack initaition, b.c.d. crack propagates as loads increases

the boundary condition (loading and supports) is such that mode I fracture is favored. As a result, the crack growth is in mode I and mode II contribution are negligible. The pattern of crack growth and load-displacement curve obtained are similar to the one obtained experimentally during mode I fracture test on three-point notched concrete beam. The load-displacement curve has been plotted for different length scale parameter in order to study its effect as shown in the Figure 4.

The Load-displacement plot shows that for the fixed value of material properties, the length scale parameter has negligible effect on the load-displacement curve. The length scale parameter in a sense controls the damage area. A higher value of b would mean that more area will get damage and so the corresponding peak load will slightly increase. In order to study the effect of mesh size on the load-displacement curve, simulations are performed at mesh size of 0.3 mm and 0.4 mm at a constant length scale parameter of 2.0 mm. The results shows that the proposed model is almost independent of the mesh size as evident in Figure 5. The mesh independency is one of the attributes of phase field model. The numerical result generally converges when a relative fine mesh is used. In this case the converging mesh size seems to be 0.4 mm.

5 Conclusion

After simulating mode I fracture in concrete, it is concluded that the phase field model can simulate the fracture in concrete. It can also produce the post-peak strain softening behavior if proper cohesive law is used. In addition, the model is mesh independent and the length scale parameter has very little effect on the peak load. The model can be used for simulating mixed mode fracture in concrete if relevant parameter related to mixed mode are taken into account.

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